Assignment 12.

This homework is due *Thursday* Dec 3.

There are total 47 points in this assignment. 42 points is considered 100%. If you go over 42 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

This assignment is longer than usual and is worth $\frac{5}{3}$ of a regular assignment in terms of the course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much. This assignment covers sections 6.2, 6.4 in Bartle–Sherbert.

(1) [3pt] (~6.2.3c) Find the points of relative extrema of the function $f(x) = x|x^2 - 12|$ for $-5 \le x \le 6$.

(2) [2pt] (6.2.6) Prove that $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$. (Hint: Apply the Mean Value theorem to sin on the interval $[x, y]$.)

(3) [2pt] (6.2.17) Let f, g be differentiable on $\mathbb{R}$ and suppose that $f(0) = g(0)$, and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$. (Hint: Apply the Mean Value Theorem to $f - g$ on $[0, x]$.)

(4) For a given function f and a point x_0, find Taylor's polynomials $P_2(x)$, $P_5(x)$, $P_{2015}(x)$ of $f(x)$ at x_0.
  (a) [2pt] $f(x) = \sin x$ at $x_0 = \pi/2$. Compare to cos at 0.
  (b) [2pt] $f(x) = \cos x$ at $x_0 = -\pi/2$. Compare to sin at 0.
  (c) [2pt] $f(x) = x^3$ at $x_0 = 2$. Compare $P_3(x), P_5(x), P_{2015}(x)$ to $f(x)$.
  (d) [2pt] $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.
  (e) [2pt] $f(x) = \frac{1}{x}$ at $x_0 = 1$. Compare to the previous item.
  (You can take for granted that $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.)

(5) [3pt] (Part of exercise 6.4.7) If $x > 0$, show that
$$\left|\sqrt[3]{1+x} - \left(1 + \frac{1}{3}x - \frac{1}{9}x^2\right)\right| \le \frac{5}{81}x^3.$$
(*Hint:* Apply Taylor's Theorem to $f(x) = \sqrt[3]{1+x}$ with $n = 2$.)

(6) (a) [2pt] Suppose $A \in \mathbb{R}$. Show that $\lim\limits_{n\to\infty} \frac{A^n}{n!} = 0$.
  Hint: take tail of this sequence that starts with $m > 2|A|$ and represent
$$\frac{A^n}{n!} = \frac{A^m}{m!} \cdot \frac{A^{n-m}}{(m+1)\cdots n}.$$
  (b) [3pt] (6.4.8) If $f(x) = e^x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x.
  (c) [3pt] (6.4.9) If $g(x) = \cos x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x.

(7) [4pt] (Part of exercise 6.4.11) If $x > 0$ and $n \in \mathbb{N}$, show that
$$\left|\ln(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}\right| < \frac{x^{n+1}}{n+1}.$$
(*Hint:* Apply Taylor's Theorem to $f(x) = \ln(1+x)$.)

— see next page —

Since a class was canceled, below we prove couple theorems that would be proved in that class.

(8) [6pt] Go through the steps below to prove **Taylor's Theorem** (refer to 6.4.1 in Textbook if you have difficulties):
Let $n \in \mathbb{N}$, let $I = [a, b]$, and let $f : I \to \mathbb{R}$ be such that f, f',..., $f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). If $x_0 \in I$ and then for any $x \in I$ there exists a point c between x and x_0 such that $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Remark. $P_n(x)$ is usually called the nth Taylor polynomial of f at x_0, $R_n(x)$ is called the remainder in the derivative (Lagrange) form.

(a) Denote $F(t) = f(x) - f(t) - (x - t)f'(t) - \ldots - \frac{(x-t)^n}{n!} f^{(n)}(t)$, where t is a point on the interval J with endpoints x, x_0. Show that

$$F'(t) = -\frac{(x - t)^n}{n!} f^{(n+1)}(t)$$

(b) Define

$$G(t) = F(t) - \left(\frac{x - t}{x - x_0}\right)^{n+1} F(x_0)$$

Show that $G(x_0) = G(x) = 0$ and find $G'(t)$.

(c) Apply Rolle's (or Mean Value) theorem to $G(t)$ with endpoints x, x_0 to get a point c such that $G'(c) = 0$. (Don't forget to explain why all conditions for Rolle's theorem or MVT are satisfied.)

(d) Solve the equality $G'(c) = 0$ for $F(x_0)$, compare with what is required to prove the theorem.

(9) [4pt]: In this problem, we prove **nth Derivative Test** (refer to 6.4.4 in Textbook if you have difficulties):
Let I be an interval, let x_0 be an interior point of I, and let $n \geq 2$. Suppose that the derivatives f', ..., $f^{(n)}$ exist and are continuous in a neighborhood of x_0 and that $f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$, but $f^{(n)} \neq 0$.

(i) If n is even and $f^{(n)} > 0$, then f has a relative minimum at x_0.

(ii) If n is even and $f^{(n)} < 0$, then f has a relative maximum at x_0.

(iii) if n is odd, then f has neither a relative minimum nor relative maximum at x_0.

(a) Find $(n - 1)$st Taylor Polynomial $P_{n-1}(x)$ of f. Find $R_{n-1}(x)$. (Remember that R_{n-1} involves nth derivative.)

(b) Explain why we can assume that $f^{(n)}(c)$ in the expression for R_{n-1} has the same sign as $f^{(n)}(x_0)$.

(c) Write $f(x) = P_{n-1}(x) + R_{n-1}(x)$ and inspect the sign of $R_{n-1}(x)$ to prove the conclusion of the theorem when:

(i) n is even and $f^{(n)}(x_0) > 0$,

(ii) n is even and $f^{(n)}(x_0) < 0$,

(iii) n is odd.

(10) [5pt] (6.4.14+) Use nth derivative test to determine whether or not $x = 0$ is a point of relative extremum of the following functions. If it is, specify whether it is a point maximum or minimum.

(a) $f(x) = x^n$, $n \in \mathbb{N}$,

(b) $f(x) = \sin x - \tan x$,

(c) $f(x) = \cos x - 1 + \frac{1}{2}x^2$.